

# The Spectrum of Quantum Caps in $PG(4, 4)$

Daniele Bartoli, Stefano Marcugini, Fernanda Pambianco

## Abstract

We prove the non existence of quantum caps of sizes 37 and 39. This completes the spectrum of quantum caps in  $PG(4, 4)$ . This also implies the non existence of linear  $[[37, 27, 4]]$  and  $[[39, 29, 4]]$ -codes. The problem of the existence of non linear quantum codes with such parameters remains still open.

## 1 Introduction

In the projective space  $PG(r, q)$  over the Galois Field  $GF(q)$ , an  $n$ -cap is a set of  $n$  points no 3 of which are collinear. An  $n$ -cap is called *complete* if it is not contained in an  $(n + 1)$ -cap. We call an  $n$ -cap an *n-quantum cap* if the code generated by its matrix is a *quantum stabilizer code*, that is:

**Definition 1.1.** A *quaternary quantum stabilizer code* is an additive quaternary code  $\mathcal{C}$  contained in its dual  $\mathcal{C}^\perp$ , where the duality is with respect to the symplectic form.

In particular:

**Definition 1.2.** A quantum code  $\mathcal{C}$  with parameters  $n, k, d$  ( $[[n, k, d]]$ -code), where  $k > 0$ , is a quaternary quantum stabilizer code of binary dimension  $n - k$  satisfying the following: any codeword of  $\mathcal{C}^\perp$  having weight at most  $d - 1$  is in  $\mathcal{C}$ .

The code is *pure* if  $\mathcal{C}^\perp$  does not contain codewords of weight  $< d$ , equivalently if  $\mathcal{C}^\perp$  has strength  $t \geq d - 1$ .

An  $[[n, 0, d]]$ -code  $\mathcal{C}$  is a self-dual quaternary quantum stabilizer code of strength  $t = d - 1$ .

For a more detailed introduction of quantum codes see in particular [1], [2], [4].

In 1999 Bierbrauer and Edel (see [7]) have proven that the maximum size of complete caps in  $PG(4, 4)$  is 41 and only two non equivalent 41-caps exist. One of them results to be quantic.

In 2003 (see [8]) they also have proven that in  $AG(4, 4)$  there exists a unique 40-cap. It results to be quantic in  $PG(4, 4)$ .

In 2008 Tonchev has constructed quantum caps of sizes 10, 12, 14–27, 29, 31, 33, 35 (see [16]), starting from the complete 41-quantum cap in  $PG(4, 4)$ .

In 2009 we have found examples of quantum caps of sizes 13, 28, 30, 32, 34, 36, 38, see [2] and [4]. Moreover we have proven that there are only two examples of non equivalent quantum caps of size 10 and five of size 12. In the same article we have proven by exhaustive search that no 11-quantum cap exists.

In this article we show that no quantum caps of sizes 37 and 39 exist. Therefore the following theorem holds:

**Theorem 1.3.** *If  $\mathcal{K} \subset PG(4, 4)$  is a quantum cap, then  $10 \leq |\mathcal{K}| \leq 41$ , with  $|\mathcal{K}| \neq 11, 37, 39$ .*

## 2 The searching algorithm

We performed an exhaustive search for quantum caps of sizes 37 and 39. To do this, programs in C/C++ have been utilized.

We start from caps, complete and incomplete, in  $PG(3, 4)$  where the classification is known (see [3] and [12]) and we try to extend every starting cap joining new points in  $PG(4, 4)$ , to obtain complete or incomplete caps of sizes 37 and 39. We utilize the following geometric characterization to reduce the number of cases to examine ([4], Theorem 3.4).

**Theorem 2.1.** *The following are equivalent:*

1. *A pure quantum  $[[n, k, d]]$ -code which is linear over  $\mathbb{F}_4$ .*
2. *A set of  $n$  points in  $PG(\frac{n-k}{2} - 1, 4)$  of strength  $t = d - 1$ , such that the intersection size with any hyperplane has the same parity as  $n$ .*
3. *An  $[n, k]_4$  linear code of strength  $t = d - 1$ , all of whose weights are even.*
4. *An  $[n, k]_4$  linear code of strength  $t = d - 1$  which is self-orthogonal with respect to the Hermitian form.*

According to the previous theorem we can consider starting caps in  $PG(3, 4)$  of odd size only.

In particular we consider in our search only caps of sizes 13, 15 and 17 in  $PG(3, 4)$ , since the following theorem and the non existence of particular linear codes.

**Theorem 2.2.** *The following are equivalent:*

1. An  $[n, k, d']_q$ -code with  $d' \geq d$ .
2. A multiset  $\mathcal{M}$  of  $n$  points of the projective space  $PG(k-1, q)$ , satisfying the following: for every hyperplane  $H \subset PG(k-1, q)$  there are at least  $d$  points of  $\mathcal{M}$  outside  $H$  (in the multiset sense).

More precisely, we know that linear codes with  $n = 37, 39$   $k = 5$  and  $d > n - 12$  do not exist (see [13]) and so there exists an hyperplane which contains at least 12 points of the caps.

Then we consider only the examples of non equivalent caps in  $PG(3, 4)$  contained in the following table:

Table 1: Number and type of non equivalent caps  $\mathcal{K} \subset PG(3, 4)$ , with  $|\mathcal{K}| = 13, 15, 17$

$ \mathcal{K} $	# COMPLETE CAPS	# INCOMPLETE CAPS
13	1	3
15	0	1
17	1	0

We proceeded in this way:

1. we start from all non equivalent caps, complete and incomplete, in an hyperplane of  $PG(4, 4)$  and we extend them by the addition of new points of  $PG(4, 4) \setminus PG(3, 4)$ ;
2. we obtain the caps of sizes 37 and 39 in  $PG(4, 4)$  by an exhaustive search;
3. we control if an obtained cap is a quantum cap, according to Theorem 2.1. In particular we check if all the weights of the linear code generated by the cap are even.

### 3 Results

We finish our search, finding no examples of quantum caps in  $PG(4, 4)$  of sizes 37 and 39. According [2], [4], [7], [8] and [16] we have proven the following:

**Theorem 3.1.** *If  $\mathcal{K} \subset PG(4, 4)$  is a quantum cap, then  $10 \leq |\mathcal{K}| \leq 41$ , with  $|\mathcal{K}| \neq 11, 37, 39$ .*

## References

- [1] D. Bartoli, Quantum codes and related geometric properties, degrees thesis 2008, University of Perugia, Italy
- [2] D. Bartoli, J. Bierbrauer, S. Marcugini and F. Pambianco, Geometric constructions of quantum codes, *submitted*.
- [3] D. Bartoli, S. Marcugini and F. Pambianco, A computer based classification of caps in  $PG(3, 4)$ , *Rapporto Tecnico* - 8/2009, Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Perugia, Italy, (2009).
- [4] D. Bartoli, S. Marcugini and F. Pambianco, New quantum caps in  $PG(4, 4)$ , *arXiv:0905.1059v2*.
- [5] C. Bennett, D. DiVincenzo, J. Smolin and W. Wootters, Mixed state entanglement and quantum error correction, *Phys. Rev. A* **54** (1996), 3824-3851.
- [6] J. Bierbrauer, Introduction to Coding Theory, CHAPMAN & HALL/CRC (2005).
- [7] J. Bierbrauer and Y. Edel, 41 is the Largest Size of a Cap in  $PG(4, 4)$ , *Designs, Codes and Cryptography* **16** (1999), 151-160.
- [8] J. Bierbrauer and Y. Edel, The largest cap in  $AG(4, 4)$  and its uniqueness, *DESI* **29** (2003), 99-104.
- [9] A. R. Calderbank, E. M. Rains, P. M. Shor and N. J. A. Sloane, Quantum error correction via codes over  $GF(4)$ , *IEEE Transactions on Information Theory* **44** (1998), 1369-1387.
- [10] A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, On the spectrum of size of complete caps in projective spaces  $PG(n, q)$  of small dimension, *Proceedings of ACCT 2008, Eleventh International Workshop on Algebraic and Combinatorial Coding Theory, PAMPOROVO, Bulgaria 16-22 June 2008*, 57-62.
- [11] A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, On sizes of complete caps in projective spaces  $PG(n, q)$  and arcs in planes  $PG(2, q)$ , *Journal of Geometry*, to appear.
- [12] G. Faina, S. Marcugini, A. Milani and F. Pambianco, The size  $k$  of the complete  $k$ -caps in  $PG(n, q)$  for small  $q$  and  $3 \leq n \leq 5$ , *Ars Combinatoria* **50** (1998), 235-243.

- [13] M. Grassl, <http://www.codetables.de>.
- [14] E. Knill and R. Laflamme, A theory of quantum error-correcting codes, *Phys. Rev. A* **55** (1997), 900-911.
- [15] B. Schumacher, Quantum coding, *Phys. Rev. A* **51** (1995), 2738-2747.
- [16] V. Tonchev, Quantum codes from caps, *Discrete Mathematics* **308** (2008), 6368-6372.